XV. *On the Thermodynamic Theory of Waves of Finite Longitudinal Disturbance.*

By W. J. Macquorn Rankine, *C.E., LL.D., F.R.SS. Lond. & Edin., &c.*

Received August 13,—Read December 16, 1869.

§ 1. The object of the present investigation is to determine the relations which must exist between the laws of the elasticity of any substance, whether gaseous, liquid, or solid, and those of the wave-like propagation of a finite longitudinal disturbance in that substance; in other words, of a disturbance consisting in displacements of particles along the direction of propagation, the velocity of displacement of the particles being so great that it is not to be neglected in comparison with the velocity of propagation. In particular, the investigation aims at ascertaining what conditions as to the transfer of heat from particle to particle must be fulfilled in order that a finite longitudinal disturbance may be propagated along a prismatic or cylindrical mass without loss of energy or change of type: the word *type* being used to denote the relation between the extent of disturbance at a given instant of a set of particles, and their respective undisturbed positions. The disturbed matter in these inquiries may be conceived to be contained in a straight tube of uniform cross-section and indefinite length.

§ 2. *Mass-Velocity.*—A convenient quantity in the present investigation is what may be termed the *mass-velocity* or *somatic velocity*—that is to say, the mass of matter through which a disturbance is propagated in a unit of time while advancing along a prism of the sectional area unity. That mass-velocity will be denoted by m.

Let S denote the *bulkiness*, or the space filled by unity of mass, of the substance in the undisturbed state, and a the linear velocity of advance of the wave; then we have evidently

$$a=m\mathrm{S}. \qquad (1)$$

§ 3. *Cinematical Condition of Permanency of Type.*—If it be possible for a wave of disturbance to be propagated in a uniform tube without change of type, that possibility is expressed by the uniformity of the mass-velocity m for all parts of the wave.

Conceive a space in the supposed tube, of an invariable length Δx, to be contained between a pair of transverse planes, and let those planes advance with the linear velocity a in the direction of propagation. Let the values of the bulkiness of the matter at the foremost and aftermost planes respectively be denoted by s_1 and s_2, and those of the velocity of longitudinal disturbance by u_1 and u_2. Then the linear velocities with which the particles traverse the two planes respectively are as follows: for the foremost plane u_1-a, for the aftermost plane u_2-a. The uniformity of type of the disturbance involves as a condition, that equal masses of matter traverse the two planes respectively in a given time, being each, in unity of time, expressed by the *mass-velocity*; hence we have, as the *cinematical condition* of uniformity of type, the following equation:

$$\frac{a-u_1}{s_1}=\frac{a-u_2}{s_2}=\frac{a}{\mathrm{S}}=m. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Another way of expressing the same condition is as follows:

$$\Delta u=-m\Delta s. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

§ 4. *Dynamical Condition of Permanency of Type.*—Let p_1 and p_2 be the intensities of the longitudinal pressure at the foremost and aftermost advancing planes respectively. Then in each unit of time the difference of pressure, p_2-p_1, impresses on the mass m the acceleration u_2-u_1, and consequently, by the second law of motion, we have the following value for the difference of pressure:

$$p_2-p_1=m(u_2-u_1). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Then substituting for the acceleration u_2-u_1 its value in terms of the change of bulkiness as given by equation (3), we obtain, for the *dynamical condition* of permanency of type, the following equation,

$$p_2-p_1=m^2(s_1-s_2), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which may also be put in the form of an expression giving the value of the square of the mass-velocity, viz.

$$m^2=-\frac{\Delta p}{\Delta s}=-\frac{dp}{ds}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The square of the linear velocity of advance is given by the following equation:

$$a^2=m^2\mathrm{S}^2=-\mathrm{S}^2\frac{dp}{ds}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The integral form of the preceding equations may be expressed as follows. Let S, as before, be the bulkiness in the undisturbed state, and P the longitudinal pressure; then in a wave of disturbance of permanent type, we must have the following condition fulfilled:

$$p+m^2s=\mathrm{P}+m^2\mathrm{S}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

§ 5. *Waves of Sudden Disturbance.*—The condition expressed by the equations of the preceding section holds for any type of disturbance, continuous or discontinuous, gradual or abrupt. To represent, in particular, the case of a single abrupt disturbance, we must conceive the foremost and aftermost advancing planes already mentioned to coalesce into one. Then P is the longitudinal pressure, and S the bulkiness, in front of the advancing plane; p is the lonigtudinal pressure, and s the bulkiness, behind the advancing plane; and the advancing plane is a wave-front of *sudden compression* or of *sudden rarefaction**

* (*Note, added* 1st *August*, 1870.) Sir WILLIAM THOMSON has pointed out to the author, that a wave of sudden rarefaction, though mathematically possible, is an unstable condition of motion; any deviation from absolute suddenness tending to make the disturbance become more and more gradual. Hence the only wave of sudden disturbance whose permanency of type is physically possible, is one of sudden compression; and this is to be taken into account in connexion with all that is stated in the paper respecting such waves.

according as p is greater or less than P. The squares of the *mass-velocity* and of the linear velocity of advance are respectively as follows:

$$m^2=\frac{p-P}{S-s}; \quad \ldots \quad (9)$$

$$a^2=m^2S^2=\frac{p-P}{S-s}.S^2. \quad \ldots \quad (10)$$

The velocity of the disturbed particles is as follows:

$$u=m(S-s)=\frac{p-P}{m}=\sqrt{(p-P).(S-s)}; \quad \ldots \quad (11)$$

and it is forward or backward according as the wave is one of compression or of rarefaction.

The energy expended in unity of time, in producing any such wave, is expressed by pu; for the wave may be conceived to be produced in a tube closed at one end by a moveable piston of inappreciable mass, to which there is applied a pressure p different from the undisturbed pressure P, and which consequently moves with the velocity u. The way in which that energy is disposed of is as follows: actual energy of the disturbance, $\frac{mu^2}{2}$; work done in altering bulkiness, $\frac{m(p+P)(S-s)}{2}$; and the equation of the conservation of energy is

$$pu=\frac{m}{2}\{u^2+(p+P)(S-s)\}. \quad \ldots \quad (11\text{A})$$

§ 6. *Thermodynamic Conditions.*—While the equations of the two preceding sections impose the constancy of the rate of variation of pressure with bulkiness during the disturbance $\left(\frac{dp}{ds}=-m^2\right)$ as an indispensable condition of permanency of type of the wave, they leave the limits of pressure and of bulkiness, being four quantities, connected by one equation only $\left(\frac{p_2-p_1}{s_1-s_2}=-\frac{dp}{ds}=m^2\right)$. Two only of those quantities can be arbitrary; therefore one more equation is required, and that is to be determined by the aid of the laws of thermodynamics.

It is to be observed, in the first place, that no substance yet known fulfils the condition expressed by the equation $\frac{dp}{ds}=-m^2=$ constant, between finite limits of disturbance, at a constant temperature, nor in a state of non-conduction of heat (called the *adiabatic* state). In order, then, that permanency of type may be possible in a wave of longitudinal disturbance, there must be both change of temperature and conduction of heat during the disturbance.

The cylindrical or prismatic tube in which the disturbance is supposed to take place being ideal, is to be considered as non-conducting. Also, the foremost and aftermost transverse advancing planes, or front and back of the wave, which contain between them the particles whose pressure and bulkiness are in the act of varying, are to be considered as non-conducting, because of there being an indefinite length of matter before the foremost and behind the aftermost plane, to resist conduction.

The transfer of heat, therefore, takes place wholly amongst the particles undergoing variation of pressure and bulkiness; and therefore for any given particle, during its passage from the front to the back of the wave, the *integral amount of heat received must be nothing*; and this is the thermodynamic condition which gives the required equation. That equation is expressed as follows:

$$\int_{\phi_1}^{\phi_2} \tau d\phi = 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

in which τ denotes absolute temperature, and ϕ the "thermodynamic function." The value of that function, as explained in various papers and treatises on thermodynamics, is given by the following formula:

$$\phi = \mathrm{J}c \text{ hyp log } \tau + \chi(\tau) + \frac{d\mathrm{U}}{d\tau}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12\text{ A})$$

in which J is the dynamical value of a unit of heat; c, the real specific heat of the substance; $\chi(\tau)$, a function of the temperature alone, which is $=0$ for all temperatures at which the substance is capable of approximating indefinitely to the perfectly gaseous state, and is introduced into the formula solely to provide for the possible existence of substances which at some temperatures are incapable of approximating to the perfectly gaseous state; and U, the work which the elastic forces in unity of mass are capable of doing at the constant temperature τ. The substitution for the integral in equation (12) of its value in terms of p and s for any particular substance, gives a relation between the limits of pressure p_1 and p_2, and the limits of bulkiness s_1 and s_2, which being combined with equation (5), or with any one of the equivalent equations (6), (8), or (9), completes the expression of the laws of the propagation of waves of finite longitudinal disturbance and permanent type in that particular substance.

§ 7. *Assumption as to Transfer of Heat.*—In applying the principles of the preceding section to the propagation of waves of longitudinal disturbance, it is obviously assumed that the transfer of heat takes place between the various particles which are undergoing disturbance at a given time, in such a manner as to ensure the fulfilment of the dynamical condition of permanency of type. It appears highly probable, that how great soever the resistance of the substance to the conduction of heat may be, that assumption as to the transfer is realized when the disturbance is *sudden*, as described in § 5; for then particles in all the successive stages of the change of pressure and bulkiness within the limits of the disturbance are at inappreciable distances from each other; so that the resistance to the transfer of heat between them is inappreciable.

But when the disturbance is not sudden, it is probable that the assumption as to the transfer of heat is fulfilled in an approximate manner only; and if such is the case, it follows that *the only longitudinal disturbance which can be propagated with absolute permanence of type is a sudden disturbance.*

§ 8. *Combination of the Dynamic and Thermodynamic Equations.*—In every fluid, and probably in many solids, the quantity of heat received during an indefinitely small change of pressure dp and of bulkiness ds is capable of being expressed in either of the following forms:

$$\frac{\tau d\phi}{\mathrm{J}}=c_s\frac{d\tau}{dp}dp+c_p\frac{d\tau}{ds}ds;$$

in which c_s and c_p denote the specific heat at constant bulkiness and at constant pressure respectively; and the differential coefficients $\frac{d\tau}{dp}$ and $\frac{d\tau}{ds}$ of the absolute temperature are taken, the former on the supposition that the bulkiness is constant, and the latter on the supposition that the pressure is constant. Let it now be supposed that the bulkiness varies with the pressure according to some definite law; and let the actual rate of variation of the bulkiness with the pressure be denoted by $\frac{ds}{dp}$. Then equation (12) may be expressed in the following form:

$$\int_{p_1}^{p_2}dp\,.\left\{c_s\frac{d\tau}{dp}+c_p\frac{d\tau}{ds}\cdot\frac{ds}{dp}\right\}=0.$$

Now, according to the dynamic condition of permanence of type, we have by equation (6),

$$\frac{ds}{dp}=-\frac{1}{m^2};$$

which, being substituted in the preceding integrals, gives the following equations from which to deduce *the square of the mass-velocity*:

$$\int_{p_1}^{p_2}dp\,.\left\{m^2c_s\frac{d\tau}{dp}-c_p\frac{d\tau}{ds}\right\}=0. \quad \ldots \quad (13)$$

It is sometimes convenient to substitute for $c_p\frac{d\tau}{ds}$ the following value, which is a known consequence of the laws of thermodynamics:

$$c_p\frac{d\tau}{ds}=c_s\frac{d\tau}{ds}+\frac{\tau dp}{\mathrm{J}d\tau}, \quad \ldots \quad (13\ \text{A})$$

the differential coefficient $\frac{dp}{d\tau}$ being taken on the supposition that s is constant. The equations (13) and (13 A) are applicable to all fluids, and probably to many solids also, especially those which are isotropic.

The determination of the squared mass-velocity, m^2, enables the bulkiness s for any given pressure p, and the corresponding velocity of disturbance u, to be found by means of the following formulæ, which are substantially identical with equations (8) and (3) respectively:

$$s=\mathrm{S}+\frac{\mathrm{P}-p}{m^2}; \quad \ldots \quad (14)$$

$$u=m(\mathrm{S}-s)=\frac{p-\mathrm{P}}{m}. \quad \ldots \quad (15)$$

Equation (15) also serves to calculate the pressure p corresponding to a given velocity of disturbance u. It may here be repeated that the linear velocity of advance is $a=m\mathrm{S}$ (equation 1).

§ 9. *Application to a Perfect Gas.*—In a perfect gas, the specific heat at constant volume, c_s, and the specific heat at constant pressure, c_p, are both constant; and consequently bear to each other a constant ratio, $\frac{c_p}{c_s}$, whose value for air, oxygen, nitrogen, and hydrogen is nearly 1·41, and for steam-gas nearly 1·3. Let this ratio be denoted by γ. Also, the differential coefficients which appear in equations (13) and (13A) have the following values:—

$$\left.\begin{aligned} \frac{d\tau}{dp}&=\frac{\tau}{p}=\frac{s}{\mathrm{J}(c_p-c_s)}=\frac{s}{\mathrm{J}(\gamma-1)c_s}; \\ \frac{d\tau}{ds}&=\frac{\tau}{s}=\frac{p}{\mathrm{J}(c_p-c_s)}=\frac{p}{\mathrm{J}(\gamma-1)c_s}; \\ \frac{dp}{d\tau}&=\frac{p}{\tau}=\frac{\mathrm{J}(c_p-c_s)}{s}=\frac{\mathrm{J}(\gamma-1)c_s}{s}. \end{aligned}\right\} \quad \ldots \ldots \ldots \quad (16)$$

When these substitutions are made in equation (13), and constant common factors cancelled, it is reduced to the following:

$$\int_{p_1}^{p_2} dp\,.\,\{m^2 s-\gamma p\}=0. \quad \ldots \ldots \ldots \quad (17)$$

But according to the dynamical condition of permanence of type, as expressed in equation (8), we have $m^2s=m^2\mathrm{S}+\mathrm{P}-p$; whence it follows that the value of the integral in equation (17) is

$$\int_{p_1}^{p_2} dp\,.\,\{m^2\mathrm{S}+\mathrm{P}-(\gamma+1)p\}=(m^2\mathrm{S}+\mathrm{P})(p_2-p_1)-\frac{\gamma+1}{2}(p_2^2-p_1^2)=0;$$

which, being divided by p_2-p_1, gives for the square of the mass-velocity of advance the following value:

$$m^2=\frac{1}{\mathrm{S}}\left\{(\gamma+1)\,.\,\frac{p_2+p_1}{2}-\mathrm{P}\right\}. \quad \ldots \ldots \ldots \quad (18)$$

The square of the linear velocity of advance is

$$a^2=m^2\mathrm{S}^2=\mathrm{S}\left\{(\gamma+1)\,.\,\frac{p_2+p_1}{2}-\mathrm{P}\right\}. \quad \ldots \ldots \quad (19)$$

The velocity of disturbance u corresponding to a given pressure p, or, conversely, the pressure p corresponding to a given velocity of disturbance, may be found by means of equation (15).

Such are the general equations of the propagation of waves of longitudinal disturbance of permanent type along a cylindrical mass of a perfect gas whose undisturbed pressure and bulkiness are respectively P and S. In the next two sections particular cases will be treated of.

§ 10. *Wave of Oscillation in a Perfect Gas.*—Let the mean between the two extreme pressures be equal to the undisturbed pressure; that is, let

$$\frac{p_2+p_1}{2}=\text{P}; \quad \ldots \quad (20)$$

then equations (18) and (19) become simply

$$m^2=\frac{\gamma\text{P}}{\text{S}}, \quad \ldots \quad (21)$$

and

$$a^2=\gamma\text{PS}; \quad \ldots \quad (22)$$

the last of which is LAPLACE'S well-known law of the propagation of sound. The three equations of this section are applicable to an indefinitely long series of waves in which equal disturbances of pressure take place alternately in opposite directions.

§ 11. *Wave of Permanent Compression or Dilatation in a Tube of Perfect Gas.*—To adapt equation (18) to the case of a wave of permanent compression or dilatation in a tube of perfect gas, the pressure at the front of the wave is to be made equal to the undisturbed pressure, and the pressure at the back of the wave to the final or permanently altered pressure. Let the final pressure be denoted simply by p; then $p_1=\text{P}$, and $p_2=p$; giving for the square of the mass-velocity

$$m^2=\frac{1}{\text{S}}\left\{(\gamma+1)\frac{p}{2}+(\gamma-1)\frac{\text{P}}{2}\right\}, \quad \ldots \quad (23)$$

for the square of the linear velocity of advance

$$a^2=m^2\text{S}^2=\text{S}\left\{(\gamma+1)\frac{p}{2}+(\gamma-1)\frac{\text{P}}{2}\right\}, \quad \ldots \quad (24)$$

and for the final velocity of disturbance

$$u=\frac{p-\text{P}}{m}=(p-\text{P})\sqrt{\left\{\frac{\text{S}}{(\gamma+1)\frac{p}{2}+(\gamma-1)\frac{\text{P}}{2}}\right\}}. \quad \ldots \quad (25)$$

Equations (23) and (24) show that a wave of condensation is propagated faster, and a wave of rarefaction slower, than a series of waves of oscillation. They further show that there is no upper limit to the velocity of propagation of a wave of condensation; and also that to the velocity of propagation of a wave of rarefaction there is a lower limit, found by making $p=0$ in equations (23) and (24). The values of that lower limit, for the squares of the mass-velocity and linear velocity respectively, are as follows:—

$$m^2(p=0)=\frac{(\gamma-1)\text{P}}{2\text{S}}; \quad \ldots \quad (26)$$

$$a^2(p=0)=\frac{(\gamma-1)\text{PS}}{2}; \quad \ldots \quad (27)$$

and the corresponding value of the velocity of disturbance, being its negative limit, is

$$u(p=0)=-\sqrt{\left\{\frac{2\text{PS}}{\gamma-1}\right\}}. \quad \ldots \quad (28)$$

It is to be borne in mind that the last three equations represent a state of matters which may be approximated to, but not absolutely realized.

Equation (25) gives the velocity with which a piston in a tube is to be moved inwards or outwards as the case may be, in order to produce a change of pressure from P to p, travelling along the tube from the piston towards the further end. Equation (25) may be converted into a quadratic equation, for finding p in terms of u; in other words, for finding what pressure must be applied to a piston in order to make it move at a given speed along a tube filled with a perfect gas whose undisturbed pressure and bulkiness are P and S. The quadratic equation is as follows:

$$p^2-\left(2\mathrm{P}+\frac{\gamma+1}{2\mathrm{S}}u^2\right)p-\frac{\gamma-1}{2}\cdot\frac{\mathrm{P}u^2}{\mathrm{S}}+\mathrm{P}^2=0;$$

and its alternative roots are given by the following formula:

$$p=\mathrm{P}+\frac{\gamma+1}{4\mathrm{S}}u^2\pm\sqrt{\left\{\frac{\gamma\mathrm{P}u^2}{\mathrm{S}}+\frac{(\gamma+1)^2u^4}{16\mathrm{S}^2}\right\}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

The sign + or − is to be used, according as the piston moves inwards so as to produce condensation, or outwards so as to produce rarefaction. Suppose, now, that in a tube of unit area, filled with a perfect gas whose undisturbed pressure and volume are P and S, there is a piston dividing the space within that tube into two parts, and moving at the uniform velocity u: condensation will be propagated from one side of the piston, and rarefaction from the other; the pressures on the two sides of the piston will be expressed by the two values of p in equation (29); and the force required in order to keep the piston in motion will be the difference of these values; that is to say,

$$\Delta p=2u\cdot\sqrt{\left\{\frac{\gamma\mathrm{P}}{\mathrm{S}}+\frac{(\gamma+1)^2u^2}{16\mathrm{S}^2}\right\}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

Two limiting cases of the last equation may be noted: first, if the velocity of the piston is very small compared with the velocity of sound, that is if $\frac{\mathrm{S}u^2}{\gamma\mathrm{P}}$ is very small, we have

$$\Delta p \text{ nearly } =2u\cdot\sqrt{\left(\frac{\gamma\mathrm{P}}{\mathrm{S}}\right)}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30\text{ A})$$

secondly, if the velocity of the piston is very great compared with the velocity of sound, that is if $\frac{\gamma\mathrm{P}}{\mathrm{S}u^2}$ is very small, we have

$$\Delta p \text{ nearly } =\frac{(\gamma+1)u^2}{2\mathrm{S}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30\text{ B})$$

§ 12. *Absolute Temperature.*—The absolute temperature of a given particle of a given substance, being a function of the pressure p and bulkiness s, can be calculated for a point in a wave of disturbance for which p and s are given. In particular, the absolute temperature in a perfect gas is given by the following well-known thermodynamic formula:

$$\tau=\frac{ps}{(\mathrm{J}c_p-c_s)}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

and if, in that formula, there be substituted the value of s in terms of p, given by equations (8) and (18) combined, we find, for the absolute temperature of a particle at which the pressure is p, in a wave of permanent type, the following value:

$$\tau = \frac{PS}{J(c - c_s)} \cdot \frac{(\gamma+1)(p_1+p_2)p - 2p^2}{(\gamma+1)(p_1+p_2)P - 2P^2}; \quad \ldots \ldots \ldots \quad (32)$$

in which the first factor $\frac{PS}{J(c_p - c_s)}$ is obviously the *undisturbed* value of the absolute temperature. For brevity's sake, let this be denoted by T.

The following particular cases may be noted. In a wave of oscillation, as defined in § 10, we have $p_1 + p_2 = 2P$; and consequently

$$\tau = T \cdot \frac{(\gamma+1)Pp - p^2}{\gamma P^2}. \quad \ldots \ldots \ldots \quad (32\text{ A})$$

In a wave of permanent condensation or rarefaction, as described in § 11, let $p_1 = P$, $p_2 = P$; then the final temperature is

$$\tau = T \cdot \frac{(\gamma+1)Pp + (\gamma-1)p^2}{(\gamma+1)Pp + (\gamma-1)P^2}.$$

§ 13. *Types of Disturbance capable of Permanence.*—In order that a particular type of disturbance may be capable of permanence during its propagation, a relation must exist between the temperatures of the particles and their relative positions, such that the conduction of heat between the particles may effect the transfers of heat required by the thermodynamic conditions of permanence of type stated in § 6.

During the time occupied by a given phase of the disturbance in traversing a unit of mass of the cylindrical body of area unity in which the wave is travelling, the quantity of heat received by that mass, as determined by the thermodynamic conditions, is expressed in dynamical units by

$$\tau d\phi.$$

The time during which that transfer of heat takes place is the reciprocal $\frac{1}{m}$ of the mass-velocity of the wave. Let $\frac{d\tau}{dx}$ be the rate at which temperature varies with longitudinal distance, and k the conductivity of the substance, in dynamical units; then the same quantity of heat, as determined by the laws of conduction, is expressed by

$$\frac{1}{m} \cdot d\left(k \frac{d\tau}{dx}\right).$$

The equality of these two expressions gives the following general differential equation for the determination of the types of disturbance that are capable of permanence:

$$m\tau d \cdot \phi = d \cdot \left(k \frac{d\tau}{dx}\right) \quad \ldots \ldots \ldots \quad (33)$$

The following are the results of two successive integrations of that differential equation:—

$$\frac{dx}{d\tau} = \frac{k}{A + m\int \tau d\phi}, \quad \ldots \ldots \ldots \quad (33\text{ A})$$

$$x = B + \int \frac{k d\tau}{A + m\int \tau d\phi}; \quad \ldots \ldots \ldots \quad (33\text{ B})$$

in which A and B are arbitrary constants. The value of A depends on the magnitude of the disturbance, and that of B upon the position of the point from which x is reckoned. In applying these general equations to particular substances, the values of τ and ϕ are to be expressed in terms of the pressure p, by the aid of the formulæ of the preceding section; when equation (33 B) will give the value of x in terms of p, and thus will show the type of disturbance required.

Our knowledge of the laws of the conduction of heat is not yet sufficient to enable us to solve such problems as these for actual substances with certainty. As a hypothetical example, however, of a simple kind, we may suppose the substance to be perfectly gaseous and of constant conductivity. The assumption of the perfectly gaseous condition gives, according to the formulæ of the preceding sections,

$$\tau=\frac{\mathrm{PS}}{(\gamma-1)\mathrm{J}c_s}\cdot\frac{(\gamma+1)(p_1+p_2)p-2p^2}{(\gamma+1)(p_1+p_2)\mathrm{P}-2\mathrm{P}^2},$$

and

$$\tau d\phi=\frac{\gamma+1}{m^2(\gamma-1)}\left\{\frac{p_2+p_1}{2}-p\right\}dp.$$

It is unnecessary to occupy space by giving the whole details of the calculation; and it may be sufficient to state that the following are the results. Let

$$p-\frac{p_1+p_2}{2}=q,$$

$$\frac{p_2-p_1}{2}=q_1;$$

then

$$\frac{dx}{dp}=\frac{dx}{dq}=\frac{k}{(\gamma+1)m\mathrm{J}c_s}\cdot\frac{(\gamma-1)(p_1+p_2)-4q}{q_1^2-q^2} \quad . \quad . \quad . \quad . \quad (34)$$

$$x=\frac{k}{(\gamma+1)m\mathrm{J}c_s}\left\{\frac{(\gamma-1)(p_1+p_2)}{2q_1}\cdot\text{hyp log}\,\frac{q_1+q}{q_1-q}+2\,\text{hyp log}\left(1-\frac{q^2}{q_1^2}\right)\right\}. \quad . \quad . \quad (34\text{ A})$$

In equation (34 A) it is obvious that x is reckoned from the point where $q=0$; that is, where the pressure $p=\frac{p_2+p_1}{2}$; a mean between the greatest and least pressures. The direction in which x is positive may be either the same with or contrary to that of the advance of the wave; the former case represents the type of a wave of rarefaction, the latter that of a wave of compression. For the two limiting pressures when $q=\pm q_1$, $\frac{dx}{dq}$ becomes infinite, and x becomes positively or negatively infinite; so that the wave is infinitely long. The only exception to this is the limiting case, when the conductivity k is indefinitely small; and then we have the following results: when $p=p_1$, or $p=p_2$, $\frac{dx}{dp}$ is infinite, and x is indefinite; and for all values of p between p_1 and p_2, $\frac{dx}{dp}$ and x are each indefinitely small. These conditions evidently represent the case of a wave of abrupt rarefaction or compression, already referred to in §§ 6 and 7.

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Received October 1,—Read December 16, 1869.

Note as to previous investigations.—Four previous investigations on the subject of the transmission of waves of finite longitudinal disturbance may be referred to, in order to show in what respects the present investigation was anticipated by them, and in what respects its results are new.

The first is that of POISSON, in the Journal de l'Ecole Polytechnique, vol. vii. Cahier 14, p. 319. The author arrives at the following general equations for a gas fulfilling MARIOTTE'S law:—

$$\frac{d\varphi}{dx}=f\left\{x-at-\frac{d\varphi}{dx}t\right\},$$

$$\frac{d\varphi}{dt}+a\frac{d\varphi}{dx}+\frac{1}{2}\cdot\frac{d\varphi^2}{dx^2}=0;$$

in which φ is the velocity-function; $\frac{d\varphi}{dx}$ the velocity of disturbance, at the time t, of a particle whose distance from the origin is x; a is the limit to which the velocity of propagation of the wave approximates when $\frac{d\varphi}{dx}$ becomes indefinitely small, viz. $\sqrt{\frac{dp_0}{d\varrho_0}}$, p_0 being the undisturbed pressure and ϱ_0 the undisturbed density; and f denotes an arbitrary function. This equation obviously indicates the quicker propagation of the parts of the wave where the disturbance is forward (that is, the compressed parts) and the slower propagation of the parts where the disturbance is backward (that is, the dilated parts).

The second is that of Mr. STOKES, in the Philosophical Magazine for November 1848, 3rd series, vol. xxxiii. p. 349, in which that author shows how the type of a series of waves of finite longitudinal disturbance in a perfect gas alters as it advances, and tends ultimately to become a series of sudden compressions followed by gradual dilatations.

The third is that of Mr. AIRY, Astronomer Royal, in the Philosophical Magazine for June 1849, 3rd series, vol. xxxiv. p. 401, in which is pointed out the analogy between the above-mentioned change of type in waves of sound, and that which takes place in sea-waves when they roll into shallow water.

The fourth, and most complete, is that of the Rev. SAMUEL EARNSHAW, received by the Royal Society in November 1858, read in January 1859, and published in the Philosophical Transactions for 1860, page 133. That author obtains exact equations for the propagation of waves of finite longitudinal disturbance in a medium in which the pressure is any function of the density; he shows what changes of type, of the kind already mentioned, must go on in such waves; and he points out, finally, that in order that the type may be permanent $\varrho^2\frac{dp}{d\varrho}\left(=-\frac{dp}{ds}\right.$ in the notation of the present paper) must be a constant quantity; being the proposition which is demonstrated in an elementary way near the beginning of the present paper. Mr. EARNSHAW regards that condition as one which cannot be realized.

The *new results*, then, obtained in the present paper may be considered to be the following:—the conditions as to transformation and transfer of heat which must be fulfilled, in order that permanence of type may be realized, exactly or approximately; the types of wave which enable such conditions to be fulfilled, with a given law of the conduction of heat; and the velocity of advance of such waves.

The *method of investigation* in the present paper, by the aid of *mass-velocity* to express the speed of advance of a wave, is new, so far as I know; and it seems to me to have great advantages in point of simplicity, enabling results to be demonstrated in a very elementary manner, which otherwise would have required comparatively long and elaborate processes of investigation.